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C Aragone† and F Zypman†

Departamento de Física, Universidad Simón Bolívar, Apartado 80659, Caracas 1080-A, Venezuela

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Abstract. We present the supercoherent states. They are introduced as the eigenstates of the supersymmetric annihilation operator of the quantum mechanical supersymmetric harmonic oscillator. They have a compact expression in terms of the standard (bosonic) coherent states. For each value of the complex parameter z we now have two linearly independent orthogonal supercoherent states. One of them has pure fermionic character and is the more classical, while the other is fully supersymmetric (i.e. the mean value of the Klein operator when the system is set in this state vanishes). There are no purely bosonic supercoherent states. The supercoherent fermionic states saturate both the Heisenberg uncertainty relation and the new entropic uncertainty recently introduced by Deutsch, while the supersymmetric ones almost (but not exactly) make it. They give a classical (or almost classical) mean value for the energy of the system and do not spread along their time evolution.

1. Introduction

Today there is no need to explain the importance of the coherent states which can be found in three different ways: as minimum uncertainty states, as eigenstates of the annihilation operator or as displacement operator coherent states (Nieto 1984).

Supersymmetry almost naturally obliges us to introduce the superspace as the intrinsically quantum mechanical refinement of ordinary (bosonic) space. It has to be asked what should be (if it exists at all) the supersymmetric generalisation of the standard coherent states, once one is aware of the existence of the supersymmetric harmonic oscillator. This question is more interesting now that the relevance of supersymmetry for interrelating spectra of different atoms and ions has been pointed out (Kostelecky and Nieto 1984).

We introduce them as the eigenstates of the corresponding (supersymmetric) annihilation operator. They constitute, for each complex z, a two-dimensional subspace having a natural orthogonal basis $\{|z_f\rangle|z_s\rangle\}$. The different states of this annihilation eigenspace do not have the same quantum mechanical (or classical) properties; their analysis singles out $\{|z_f\rangle, |z_s\rangle\}$ as the two exclusively physically relevant states, as will be shown in §§ 3 and 4.

The two kinds of uncertainties (the standard and the recently introduced entropic one) (Deutsch 1983, Partovi 1983) related to the observables x and p are given in § 3. Their explicit value for $|z_f\rangle$ and $|z_s\rangle$ is obtained. In § 4 we analyse the time evolution of the different elements of the supersymmetric annihilation subspace.

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In the next section, in order to make the paper self-contained, a quick review of the superspace formulation of $N=1$ quantum mechanics is presented, leading for a particular choice of the superpotential to the one-dimensional supersymmetric harmonic oscillator.

Finally we dedicate the last section to making some comments and discussing the results presented in this paper.

2. Review of $N=1$, $d=1$ superspace quantum mechanics

The supersymmetric harmonic oscillator is a particular case, for a specific choice of the superpotential, of the $N=1$ supersymmetric quantum mechanical model presented by Salomonson and van Holten (1982) and later also considered by Cooper and Freedman (1983). A conceptually careful analysis of this quantum mechanical supersymmetric system can be found in De Witt (1984).

Superspace consists, in this case, of the points $\{t_1, \theta, \bar{\theta}\}$ having one real bosonic coordinate and two complex Grassmann variables θ, $\bar{\theta}=\theta^+$, such that

$$[\theta, t]=[\bar{\theta}, t]=0 \qquad \theta^2=\bar{\vartheta}^2=0=(\vartheta, \bar{\theta}). \tag{1a}$$

The $d=1$ metric is determined by $\eta^{00}=-1$. There are two spinorial charges q, $\bar{q}$ whose anticommutator gives the one-dimensional translation operator

$$(q, \bar{q})_+=2p^0=-2p_0=2\mathrm{i}\partial_t. \tag{1b}$$

Superfields in the physical representation are defined in the standard way (Ferrara *et al* 1974)

$$\phi(t, \theta, \bar{\theta})\equiv\exp(\mathrm{i}\bar{q}\bar{\theta}+\mathrm{i}\theta q-\mathrm{i}tp_0) \tag{2a}$$

transforming under a pure supersymmetry $g_\zeta\equiv\exp(\mathrm{i}\zeta q+\mathrm{i}\bar{q}\bar{\zeta}]$ according to

$$(g_\zeta\phi)(t, \theta, \bar{\theta})=\phi(t+\mathrm{i}\zeta\bar{\theta}-\mathrm{i}\vartheta\bar{\zeta}, \vartheta+\zeta, \bar{\theta}+\bar{\zeta}) \tag{2b}$$

or equivalently, if g_ζ is near the identity, giving rise to the infinitesimal form

$$\delta_\zeta\phi\equiv(\zeta q+\bar{q}\bar{\zeta})\phi \qquad q\equiv\partial_\theta+\mathrm{i}\bar{\theta}\partial_t \qquad \bar{q}\equiv-\partial_{\bar{\theta}}-\mathrm{i}\vartheta\partial_t. \tag{2c}$$

The associated spinorial derivatives turn out to be, in the same (symmetric) representation,

$$D\equiv-\partial_\theta+\mathrm{i}\bar{\theta}\partial_t \qquad \bar{D}\equiv\partial_{\bar{\theta}}-\mathrm{i}\theta\partial_t. \tag{3a}$$

They constitute a representation of the 1D supersymmetric algebra (1b):

$$(D, \bar{D})_+=2p^0 \qquad D^2=0=\bar{D}^2 \tag{3b}$$

anticommuting with $(q\bar{q})$. The quantum mechanical model is constructed by means of a Hermitian (real) superfield $\phi=\phi^+$:

$$\phi(t, \theta, \bar{\theta})=x(t)+\mathrm{i}\theta\psi(t)-\mathrm{i}\bar{\psi}(t)\bar{\theta}+\lambda(t)\vartheta\bar{\theta}. \tag{4}$$

It is convenient to introduce the supersymmetric momentum operator $p^1=p_1\equiv 2^{-1}[D, \bar{D}]$. Together with (1$b$) we obtain

$$D\bar{D}=p^0+p^1 \qquad \bar{D}D=p^0-p^1 \qquad (p^1)^2=(p^0)^2 \tag{5a}$$

and the useful relationships

$$p_1D=-Dp_1=p^0D \qquad p_1\bar{D}=-\bar{D}p_1=-p^0\bar{D}. \tag{5b}$$

The components of ϕ can be obtained by taking spinorial derivatives at $\theta=\bar\theta=0$ (Gates *et al* 1983)

$$\phi|_{\theta=0=\bar\theta}=x \qquad D\phi|=-\mathrm{i}\psi \qquad \bar D\phi=\mathrm{i}\bar\psi \qquad p_1\phi|=\lambda. \tag{6a}$$

Actions in this 3D superspace can be built on by integration in superspace:

$$I\sim\int\Psi(t,\vartheta,\bar\theta)\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\bar\theta\equiv\int\Psi\,\mathrm{d}^3z=\int\mathrm{d}t(p_1\phi|) \tag{6b}$$

using the fact that for one Grassmann variable $\int\theta\,\mathrm{d}\theta=+1=\int\mathrm{d}\bar\theta\,\bar\theta$.

The general action has the superspace form

$$I(\phi)\equiv\int\mathrm{d}t\,\mathrm{d}\vartheta\,\mathrm{d}\bar\theta(\tfrac{1}{2}\phi p_1\phi-f(\phi)). \tag{7a}$$

Its component expression is immediately obtained by substitution of $\int\mathrm{d}\theta\,\mathrm{d}\bar\theta\to p_1(\cdot)|$:

$$\begin{aligned}I(\phi)&=\int\mathrm{d}t[p_1(\tfrac{1}{2}\phi p_1\phi)|-p_1f(\phi)|]\\&=\tfrac{1}{2}\langle\lambda^2-x\ddot x+\mathrm{i}\bar\psi\dot\psi+\mathrm{i}\psi\dot{\bar\psi}\rangle-\langle\tfrac{1}{2}f''(x)[\psi,\bar\psi]+\lambda f'(x)\rangle.\end{aligned} \tag{7b}$$

Taking into account the infinitesimal form of the supersymmetry transformation (2c), recalling the definition of the components fields as spinorial derivatives at vanishing θ (6a) and realising that (comparing definitions (2c) and (3c)), again at vanishing θ,

$$q|=-D| \qquad \bar q|=-\bar D| \tag{8a}$$

one obtains the supersymmetric transformation laws

$$\delta_\zeta x=\delta_\zeta\phi|=(\zeta q+\bar q\bar\zeta)\phi|=-\zeta D\phi|-\bar D\phi|\bar\zeta=\mathrm{i}\zeta\psi-\mathrm{i}\bar\psi\bar\zeta \tag{8b}$$

$$\delta_\zeta\psi=-(\dot x+\mathrm{i}\lambda)\bar\zeta \qquad \delta_\zeta\lambda=\zeta\dot\psi+\dot{\bar\psi}\bar\zeta. \tag{8c}$$

Instead of the creation-annihilation fermionic variables $\psi,\bar\psi$ one can introduce into the action (7b) the Hermitian variables ψ_1, ψ_2:

$$\psi\equiv2^{-1/2}(\psi_1-\mathrm{i}\psi_2) \qquad \bar\psi\equiv2^{-1/2}(\psi_1+\mathrm{i}\psi_2) \qquad \psi_{1,2}^{+}=\psi_{1,2}. \tag{9}$$

Independent variations of λ give its value $\lambda=f'(x)$ which can be inserted into (7b) in order to obtain the expression

$$I=\langle p\dot x+\tfrac{1}{2}\mathrm{i}\psi_1\dot\psi_1+\tfrac{1}{2}\mathrm{i}\psi_2\dot\psi_2-\tfrac{1}{2}p^2-\tfrac{1}{2}p^2-\tfrac{1}{2}f'(x)^2-\tfrac{1}{2}\mathrm{i}f''(x)[\psi_1,\psi_2]\rangle \tag{10a}$$

showing that the Hamiltonian is

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}f'(x)^2+\tfrac{1}{2}\mathrm{i}f''(x)[\psi_1,\psi_2]. \tag{10b}$$

In the coordinate representation the canonical commutation relations are

$$[x,p]=\mathrm{i} \qquad (\psi_\alpha,\psi_\beta)_+=\delta_{\alpha\beta} \qquad \alpha,\beta=(1,2). \tag{11a}$$

Consequently for the creation-annihilation operators

$$(\bar\psi,\psi)_+=1 \qquad (\psi,\psi)=0=(\bar\psi,\bar\psi) \tag{11b}$$

in addition to $[\psi_1,\psi_2]=\mathrm{i}[\bar\psi,\psi]$.

In this representation fermions $\psi, \bar{\psi}$ can be represented by the time-dependent two-dimensional matrices

$$\psi(t)=\begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix} \exp(\mathrm{i}\varphi(t)) \qquad \bar{\psi}(t)=\begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \end{pmatrix} \exp(-\mathrm{i}\varphi(t)) \tag{11c}$$

where $\phi(t)$ satisfies

$$\dot{\varphi}(t)=f''(x) \tag{11d}$$

(for the supersymmetric harmonic oscillator $\phi(t)=\omega t$). The fermionic number N_{f} having eigenvalues (0, 1) turns out to be

$$N_{\mathrm{f}} \equiv \bar{\psi}\psi=\begin{pmatrix} 1 & \cdot \\ \cdot & \cdot \end{pmatrix}. \tag{11e}$$

In this case wavefunctions become two component objects $\Psi(x) \equiv (\phi_1(x), \phi_2(x))^{\mathrm{T}}$ and the Hamiltonian is

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}f'(x)^2-\tfrac{1}{2}f''(x)\sigma_3. \tag{10c}$$

The supersymmetric harmonic oscillator corresponds to choosing $f(x)=\frac{1}{2}\omega x^2$. Its Hamiltonian will be

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2 x^2-\tfrac{1}{2}\omega\sigma_3 \equiv H_{\mathrm{b}}-\tfrac{1}{2}\omega\sigma_3. \tag{10d}$$

In the natural units we are employing, $x \sim L^{1/2}$, $\omega \sim M$.

The eigenstates of this system can be labelled by their energy E and their fermionic number, 1 or 0, according to their behaviour with respect to the fermionic number operator N_{f}. (Instead of considering the N_{f} which commutes with H, the Klein operator $K \equiv (-1)^{N_{\mathrm{f}}}=-\sigma_3$ which also commutes with H can be taken.)

Since

$$N_{\mathrm{f}}\begin{pmatrix} 1 \\ \cdot \end{pmatrix}=1\begin{pmatrix} 1 \\ \cdot \end{pmatrix} \qquad N_{\mathrm{f}}\begin{pmatrix} \cdot \\ 1 \end{pmatrix}=0\begin{pmatrix} \cdot \\ 1 \end{pmatrix} \tag{11f}$$

any state $(\phi_1(x), \cdot)^{\mathrm{T}}$ has a pure fermionic character while those of the form $(\cdot, \phi_2(x))^{\mathrm{T}}$ are purely bosonic.

In the following we introduce the notation $|n\rangle$ for the unitary eigenstates of the standard harmonic oscillator:

$$H_{\mathrm{b}}|n\rangle=(n+\tfrac{1}{2})\omega|n\rangle. \tag{12a}$$

The associated (not normalised) coherent states (Klauder and Sudarshan 1968) are

$$|z\rangle \equiv \sum_{n=0}^{\infty} (n!)^{-1/2} z^n |n\rangle. \tag{12b}$$

It is straightforward to realise what the energy eigenstates and their corresponding eigenvalues of the supersymmetric Hamiltonian are (10d)

$$E_n=n\omega \qquad \Psi_{n>0}=\alpha\begin{pmatrix} |n\rangle \\ \cdot \end{pmatrix}+\beta\begin{pmatrix} \cdot \\ |n-1\rangle \end{pmatrix} \qquad \Psi_0=\begin{pmatrix} |0\rangle \\ \cdot \end{pmatrix} \qquad E_0=0. \tag{13}$$

For each non-vanishing natural n the corresponding eigenspace is two-dimensional.

When the energy vanishes there is no degeneracy, indicating that supersymmetry is not spontaneously broken.

It is worth recalling the standard definitions of creation and annihilation operators for the bosonic system H_b, as well as their commutation relations

$$a \equiv (2\omega)^{-1/2}(\mathrm{i}p + \omega x) \qquad a^+ \equiv (2\omega)^{-1/2}(-\mathrm{i}p + \omega x). \tag{14a, b}$$

Then

$$[x, p] = \mathrm{i} \qquad [a, a^+] = 1. \tag{14c, d}$$

Their inversion gives for the position and momentum operators

$$x = (2\omega)^{-1/2}(a^+ + a) \qquad p = \mathrm{i}(\omega/2)^{1/2}(a^+ - a). \tag{14e, f}$$

In terms of these operators the full Hamiltonian (10d) can be written

$$H = \omega\begin{pmatrix} a^+a & \cdot \\ \cdot & aa^+ \end{pmatrix} = \omega q^2 \tag{15a}$$

where the spinorial square root of H turns out to be the Hermitian operator

$$q \equiv \begin{pmatrix} \cdot & \mathrm{i}a^+ \\ -\mathrm{i}a & \cdot \end{pmatrix} = q^+. \tag{15b}$$

Now having described the supersymmetric harmonic oscillator we can go further and present the associated coherent states.

3. Supercoherent states

An important operator because of its supersymmetric meaning is the supersymmetric annihilation operator

$$A \equiv a \cdot 1_F + 1_B \cdot \bar{\psi} = \begin{pmatrix} a & 1 \\ \cdot & a \end{pmatrix}. \tag{16}$$

It satisfies

$$[A, H] = \omega A \qquad [A, A^+] = 2N_f \tag{17a, b}$$

the first equation being similar to the well known relation $[a, H_b] = \omega a$ which holds for the harmonic oscillator. From this equation it is straightforward to show that, if Ψ_n belongs to the energy eigenspace $E_n = n\omega$, then $|A\Psi_n\rangle$ belongs to the E_{n-1} subspace for $n \geqslant 1$.

This can also be seen directly since

$$\begin{aligned} A|\Psi_n\rangle &= \alpha A\begin{pmatrix} |n\rangle \\ \cdot \end{pmatrix} + \beta A\begin{pmatrix} \cdot \\ |n-1\rangle \end{pmatrix} \\ &= (\alpha n^{1/2} + \beta)\begin{pmatrix} |n-1\rangle \\ \cdot \end{pmatrix} + \beta(n-1)^{1/2}\begin{pmatrix} \cdot \\ |n-2\rangle \end{pmatrix}. \end{aligned} \tag{18}$$

The supercoherent states $|Z\rangle$ are defined as the eigenstates of A,

$$A|Z\rangle = z|Z\rangle. \tag{19}$$

In order to solve this equation we expand $|Z\rangle$ in terms of the natural basis $\{|0\rangle, |f_n\rangle \equiv (|n\rangle, \cdot)^T, |b_n\rangle \equiv (\cdot, |n-1\rangle)^T\}$

$$|Z\rangle = a_0|0\rangle + \sum_{n=1}^{\infty} a_n|f_n\rangle + \sum_{n=1}^{\infty} c_n|b_n\rangle. \tag{20a}$$

After insertion into (19) we find that (a_n, c_n) are determined by

$$c_{n+1} = c_1 \frac{z^n}{\sqrt{n!}} \qquad a_n = \frac{1}{\sqrt{n!}}(a_0 z^n - c_1 n z^{n-1}) \tag{20b}$$

leading to

$$|Z\rangle = a_0 |z_f\rangle + c_1 |\tilde{z}_s\rangle \tag{21a}$$

where

$$|z_f\rangle \equiv \begin{pmatrix} |z\rangle \\ \cdot \end{pmatrix} \qquad |\tilde{z}_s\rangle \equiv 2^{-1/2} \begin{pmatrix} -|z'\rangle \\ |z\rangle \end{pmatrix} \tag{21b}$$

$|z'\rangle$ meaning $\partial/\partial z\{|z\rangle\}$, $|z\rangle$ denoting the standard bosonic coherent states (12*b*). From $a|n\rangle = \sqrt{n}|n-1\rangle$, $a^+|n\rangle = (n+1)^{1/2}|n+1\rangle$ one can calculate how $|z\rangle$, $|z'\rangle$ transform under the action of (a, a^+) and their respective norms.

It is immediately obvious that

$$a|z\rangle = z|z\rangle \qquad a^+|z\rangle = |z'\rangle \qquad a|z'\rangle = |z\rangle + z|z'\rangle \tag{22a}$$

$$a^+|z'\rangle = |z''\rangle \equiv \frac{\partial^2}{\partial z^2}|z\rangle \qquad (a^+)^p|z\rangle = \frac{\partial p}{\partial z^p}|z\rangle = |z(p)\rangle \tag{22b}$$

$$\langle z|z\rangle = \exp(|z|^2) \qquad \langle z|z'\rangle = \frac{\partial}{\partial z}\{\langle z|z\rangle\} = \bar{z}\exp(|z|^2) \tag{22c}$$

$$\langle z'|z\rangle = \overline{\langle z|z'\rangle} = \frac{\partial}{\partial \bar{z}}\{\langle z|z\rangle\} = z\exp(|z|^2) \tag{22d}$$

$$\langle z'|z'\rangle = \frac{\partial}{\partial \bar{z}}\langle z|z'\rangle = (1+|z|^2)\exp(|z|^2) \tag{22e}$$

$$\langle z|z''\rangle = \frac{\partial^2}{\partial z^2}\langle z|z\rangle = \bar{z}^2\exp(|z|^2) \tag{22f}$$

$$\langle z'|z''\rangle = \frac{\partial}{\partial \bar{z}}\frac{\partial^2}{\partial z^2}\{\langle z|z\rangle\} = \bar{z}(2+|z|^2)\exp(|z|^2). \tag{22g}$$

It is convenient to introduce the new set of states $\{|z_s\rangle\}$ which belong to the supercoherent two-dimensional subspace (21*a*)

$$|z_s\rangle \equiv \frac{\bar{z}}{\sqrt{2}}|z_f\rangle + |\tilde{z}_s\rangle = \frac{1}{\sqrt{z}}\begin{pmatrix} \bar{z}|z\rangle - |z'\rangle \\ |z\rangle \end{pmatrix}. \tag{23a}$$

They are orthogonal to $|z_f\rangle$

$$\langle z_f|z_s\rangle = 0 \tag{23b}$$

and have the same norm as $|z_f\rangle$:

$$\langle z_f|z_f\rangle = \exp(|z|^2) = \langle z|z\rangle = \langle z_s|z_s\rangle. \tag{23c}$$

Any other vector belonging to the two-dimensional coherent subspace (21*a*) can be expanded as a function of $\{|z_f\rangle, |z_s\rangle\}$:

$$|Z\rangle = \alpha|z_f\rangle + \beta|z_s\rangle \qquad \langle Z|Z\rangle = (|\alpha|^2 + |\beta|^2)\exp(|z|^2). \tag{23d}$$

Due to its very simple form, $|z_f\rangle$ has the same properties its bosonic constituent $|z\rangle$ has in the sense that

$$\langle x\rangle_f \equiv \frac{\langle z_f|x|z_f\rangle}{\langle z_f|z_f\rangle} = \frac{1}{\sqrt{2}}(z+\bar{z}) = \sqrt{2}\,\mathrm{Re}\,z \tag{24a}$$

$$\langle p\rangle_f \equiv \sqrt{2}\,\mathrm{Im}\,z \tag{24b}$$

$$(\Delta x)_f^2 \equiv \langle x^2\rangle_f - \langle x\rangle_f^2 = \tfrac{1}{2} \tag{24c}$$

$$(\Delta p)_f^2 \equiv \langle p^2\rangle_f - \langle p\rangle_f^2 = \tfrac{1}{2} \qquad (\Delta x)_f(\Delta p)_f = \tfrac{1}{2}. \tag{24d, e}$$

These supercoherent states $|z_f\rangle$ constitute the more classical-like states of the ordinary harmonic oscillator in the following sense.

Let us consider the position and momentum operators x, p as given in (14e) and (14f), where for the supersymmetric system a, a^+ abbreviates

$$a\cdot 1_F = \begin{pmatrix} a & \cdot \\ \cdot & a \end{pmatrix} \qquad a^+\cdot 1_F = \begin{pmatrix} a^+ & \cdot \\ \cdot & a^+ \end{pmatrix}. \tag{25a, b}$$

Commutators with H have the respective values

$$[a, H] = \omega a \qquad [a^+, H] = -\omega a^+. \tag{25c, d}$$

The time evolution laws of the mean values of a, a^+ can thus be found immediately:

$$\langle a\rangle(t) = \langle a\rangle_0 \exp(-\mathrm{i}\omega t) \qquad \langle a^+\rangle(t) = \langle a^+\rangle_0 \exp(\mathrm{i}\omega t) \tag{25e}$$

and consequently, after definitions (14e) and (14f), it is possible to write down the time evolution of position and momentum mean values:

$$\langle x\rangle(t) = (2\omega)^{-1/2}[\langle a\rangle_0 \exp(-\mathrm{i}\omega t) + \langle a^+\rangle_0 \exp(\mathrm{i}\omega t)] \tag{26a}$$

$$\langle p\rangle(t) = \mathrm{i}(\omega/2)^{1/2}[\langle a^+\rangle_0 \exp(\mathrm{i}\omega t) - \langle a\rangle_0 \exp(-\mathrm{i}\omega t)]. \tag{26b}$$

These solutions have to be compared with the exact solutions $x_c(t)$, and $p_c(t)$ of the classical harmonic oscillator defined by $H_b = 2^{-1}p^2 + 2^{-1}\omega^2 x^2$,

$$x_c(t) = (2\omega)^{-1/2}[z_0 \exp(-\mathrm{i}\omega t) + \bar{z}_0 \exp(\mathrm{i}\omega t)] \qquad z_0 \equiv (2\omega)^{-1/2}(\mathrm{i}p_0 + \omega x_0) \tag{27a, b}$$

$$p_c(t) = \mathrm{i}(\omega/2)^{1/2}[\bar{z}_0 \exp(\mathrm{i}\omega t) - z_0 \exp(-\mathrm{i}\omega t)] \tag{27c}$$

where the classical energy is

$$H_b = \omega |z_0|^2. \tag{27d}$$

Comparing evolution laws (26) and (27) we can say that the system defined by H will be in a classical state $|\Psi_{cl}\rangle$ iff

$$\langle a\rangle_0 = \langle \Psi_{cl}|a\Psi_{cl}\rangle = z_0 \tag{27e}$$

$$\langle H\rangle_0 \equiv \langle \Psi_{cl}|H|\Psi_{cl}\rangle = \omega |z_0|^2. \tag{27f}$$

In terms of its two components $|\Psi_{cl}\rangle$ can be written

$$|\Psi_{cl}\rangle = (\Psi_{cl}^f, \Psi_{cl}^b)^T. \tag{28}$$

One has that (27f) amounts to

$$\omega(\|a|\Psi_{cl}^f\rangle\|^2 + \|a|\Psi_{cl}^b\rangle\|^2 + \|\Psi_{cl}^b\|^2) = \omega |z_0|^2. \tag{29a}$$

If one considers the operator $B \equiv (a - z_0) \cdot 1_F$, conditions (27*e*) and (27*f*) imply that

$$0 \leqslant \langle \Psi_{cl} | B^+ B \Psi_{cl} \rangle = \omega (\| a | \Psi^f_{cl} \rangle \|^2 + \| a | \Psi^b_{cl} \rangle \|^2 - |z_0|^2). \tag{29b}$$

Introducing (29*a*) into this last inequality one ends up with

$$0 \leqslant \| B | \Psi_{cl} \rangle \|^2 = -\omega \| \Psi^b_{cl} \|^2 \leqslant 0 \tag{29c}$$

leading to

$$\| B | \Psi_{cl} \rangle \| = 0 = \| \Psi^b_{cl} \| \tag{29d}$$

for the classical state of the supersymmetric harmonic oscillator. Equation (29*d*) can be quickly solved

$$|\Psi_{cl}\rangle = (\Psi^f_{cl}, \cdot)^T \tag{30a}$$

where the non-vanishing component has to obey the eigenvalue equation

$$B \begin{pmatrix} |\Psi^f_{cl}\rangle \\ \cdot \end{pmatrix} = (a - z_0) \cdot 1_F \begin{pmatrix} |\Psi^f_{cl}\rangle \\ \cdot \end{pmatrix} = 0 \Rightarrow |\Psi^f_{a}\rangle = |z_0\rangle \tag{30b}$$

whose solution evidently is the supercoherent state $|z_f\rangle$ defined in (21*b*), i.e. the most classical state of the harmonic oscillator is the supersymmetric fermionic state $|z_f\rangle$.

The same mean values (24) can be calculated when the system is in the supercoherent state $|z_s\rangle = 2^{-1/2}(\bar{z}|z\rangle - |z'\rangle, |z\rangle)^T$:

$$\| z_s \|^{-2} \langle z_s | x | z_s \rangle = \sqrt{2}\, \mathrm{Re}\, z \tag{31a}$$

$$\| z_s \|^{-2} \langle z_s | p | z_s \rangle = \sqrt{2}\, \mathrm{Im}\, z. \tag{31b}$$

Moreover, using the algebra (22) it can be seen that

$$\langle x^2 \rangle_s \equiv \langle z_s | x^2 | z_s \rangle \| z_s \|^{-2} = 1 + 2(\mathrm{Re}\, z)^2 \tag{32a}$$

$$\langle p^2 \rangle_s \equiv \| z_s \|^{-2} \langle z_s | p^2 | z_s \rangle = 1 + 2(\mathrm{Im}\, z)^2. \tag{32b}$$

Consequently

$$(\Delta x)^2_s \equiv \langle x^2 \rangle_s - \langle x \rangle^2_s = 1 \tag{33a}$$

$$(\Delta p)^2_s \equiv \langle p^2 \rangle_s - \langle p \rangle^2_s = 1. \tag{33b}$$

They have a uniform spread, even though it is not the minimal one (24*c*) and (24*d*).

Their uncertainty becomes unity:

$$(\Delta x)_s (\Delta p)_s = 1. \tag{33c}$$

Now we explore the results corresponding to choosing an arbitrary vector (21*a*) belonging to the two-dimensional space spanned by $\{|z_f\rangle, |z_s\rangle\}$.

Let $|z\rangle$ be given in the form

$$|z\rangle = \sin\theta\, e^{i\varphi} |z_f\rangle + \cos\theta |z_s\rangle. \tag{34}$$

The mean values of $\{x, x^2, p, p^2\}$ when the system is in the $|z\rangle$ state can be obtained immediately:

$$\langle x \rangle_z = \| z \|^{-2} \langle z | x | z \rangle = \sqrt{2}\, \mathrm{Re}\, z - \tfrac{1}{2} \cos\varphi \sin 2\theta \tag{35a}$$

$$\langle p \rangle_z = \| z \|^{-2} \langle z | p | z \rangle = \sqrt{2}\, \mathrm{Im}\, z - \tfrac{1}{2} \sin\varphi \sin 2\vartheta \tag{35b}$$

$$\langle x^2 \rangle_z = 1 + 2(\mathrm{Re}\, z)^2 - \tfrac{1}{2} \sin^2\theta - \sqrt{2}(\mathrm{Re}\, z) \cos\varphi \sin 2\theta \tag{35c}$$

$$\langle p^2 \rangle_z = 1 + 2(\mathrm{Im}\, z)^2 - \tfrac{1}{2} \sin^2\theta - \sqrt{2}(\mathrm{Im}\, z) \sin\varphi \sin 2\theta \tag{35d}$$

which for $\theta = 0$ reduce to expressions (31) and (32).

It happens again that $(\Delta x)^2$ and $(\Delta p)^2$ do not depend upon z. They turn out to be

$$(\Delta x)^2_z = 1 - \tfrac{1}{2}\sin^2\theta - \tfrac{1}{4}\cos^2\varphi \sin^2 2\theta \tag{35e}$$

$$(\Delta p)^2_z = 1 - \tfrac{1}{2}\sin^2\theta - \tfrac{1}{4}\sin^2\varphi \sin^2 2\theta. \tag{35f}$$

The analysis of

$$f(\theta, \phi) \equiv \tfrac{1}{2}\sin^2\theta + \tfrac{1}{4}\cos^2\phi \sin^2 2\theta \equiv 4^{-1}[1 - \cos 2\theta + \cos^2\phi(1 - \cos^2 2\theta)]$$

shows that $0 \leqslant f(\theta, \varphi) \leqslant \frac{9}{16}$.

Therefore, each uncertainty is bounded between $\frac{7}{16}$ and 1, their product having the exact value

$$(\Delta x)^2_z(\Delta p)^2_z = \tfrac{1}{4}(1 + \cos^4\theta + 2\cos^6\theta + \sin^2 2\varphi \sin^2\theta \cos^4\theta) \tag{36}$$

which can be shown to be bounded between $\frac{1}{4}$ and 1, the values corresponding to $|z_f\rangle$ and $|z_s\rangle$ respectively.

The fact that $|z_f\rangle$ is a pure fermionic state is reflected through the value of $\langle K\rangle_f$

$$\langle K\rangle_f = -1 \tag{37a}$$

while $|z_s\rangle$ is a pure mixture of fermionic and bosonic states since

$$\langle K\rangle_s = 0. \tag{37b}$$

Using the fermionic number operator N_f one would have equivalently found

$$\langle N_f\rangle_f = 1 \qquad \langle N_f\rangle_s = \tfrac{1}{2}. \tag{37c, d}$$

Recently, Deutsch has proposed a more appropriate definition of the uncertainty using the concept of quantum entropy $S_A(|\psi\rangle)$ of an observable A when the system is in the state $|\psi\rangle$

$$S_A(|\psi\rangle) = -\sum_a |\langle a|\psi\rangle|^2 \ln|\langle a|\psi\rangle|^2 \tag{38}$$

(the sum is over the whole set of A eigenstates $A|a\rangle = \alpha|a\rangle$).

Let us first consider the supercoherent pure fermionic states $|z_f\rangle'$ and the set of operators (x, p). Since by $(21b)$

$$|\hat{z}_f\rangle = \sum_{n=0}^{\infty} (n!)^{-1/2} z^n |n\rangle \begin{pmatrix}1\\0\end{pmatrix} \exp(-|z|^2/2)$$

we have

$$\langle x|\hat{z}_f\rangle = \sum_{n=0}^{\infty} (n!)^{-1/2} z^n \langle x|n\rangle \exp(-|z|^2/2)$$

$$= \exp(-|z|^2/2) \sum_{n=0}^{\infty} (n!)^{-1/2} z^n \frac{1}{2^{n/2}\pi^{1/4}} \exp(-x^2/2)(n!)^{-1/2} H_n(x) \tag{39a}$$

$$\langle p|\hat{z}_f\rangle = \sum_{n=0}^{\infty} (n!)^{-1/2} z^n \langle p|n\rangle \exp(-|z|^2/2)$$

$$= \sum_{n=0}^{\infty} (n!)^{-1/2} z^n \exp(-|z|^2/2)(-\mathrm{i})^n \frac{1}{2^{n/2}\pi^{1/4}} (n!)^{-1/2} \exp(-p^2/2) H_n(p) \tag{39b}$$

where $H_n(x)$ and $H_n(p)$ are the Hermite polynomials. Recalling the Rodrigues' expression for these polynomials it can be seen that

$$\langle p|\hat{z}_f\rangle = \pi^{-1/4} \exp(-|z|^2/2) \exp(p^2/2) \exp[-(p + \mathrm{i}z/\sqrt{2})^2]. \tag{40a}$$

Therefore

$$|\langle p|\hat{z}_f\rangle|^2 = \pi^{-1/2}\exp[-(p-\sqrt{2}\,\mathrm{Im}\,z)^2] \tag{40b}$$

and consequently the entropy of the observable p when the system is in the unitary state $|\hat{z}_f\rangle \equiv \|z_f\|^{-1}\,|z_f\rangle$ has the value

$$\begin{aligned} S_p(|\hat{z}_f\rangle) &= \int_{-\infty}^{+\infty} \pi^{-1/2}\exp[-(p-\sqrt{2}\,\mathrm{Im}\,z)^2\ln\{\pi^{-1/2}\exp[-(p-\sqrt{2}\,\mathrm{Im}\,z)^2]\}\,\mathrm{d}p \\ &= \tfrac{1}{2}+\tfrac{1}{2}\ln\pi. \end{aligned} \tag{40c}$$

The entropy of the position operator x when the system is set in the state $|\hat{z}_f\rangle$ can be computed in a similar way from (15a). One obtains

$$S_x(|\hat{z}_f\rangle) = \tfrac{1}{2}+\tfrac{1}{2}\ln\pi. \tag{40d}$$

Therefore the uncertainty in $\{x, p\}$ when the system is in the state $|\hat{z}_f\rangle$ has the value

$$U(x, p, |\hat{z}_f\rangle) \equiv S_x(|\hat{z}_f\rangle) + S_p(|\hat{z}_f\rangle) = 1+\ln\pi \tag{40e}$$

i.e. the supercoherent states $|z_f\rangle$ saturate the entropic bound conjectured some time ago by Everett (1973) and proved later on independently by Beckner (1975) and Bialynicki-Birula and Mycielski (1975).

In the case of the $|z_s\rangle$ supercoherent states an orthonormal basis for the position operator is

$$|x_f\rangle \equiv \begin{pmatrix} |x\rangle \\ \cdot \end{pmatrix} \qquad |x_b\rangle = \begin{pmatrix} \cdot \\ |x\rangle \end{pmatrix}. \tag{41}$$

Taking into account the expression (23a) and again using the Rodrigues' form for the Hermite polynomials we have that

$$\langle x_f|\hat{z}_s\rangle|^2 = \frac{1}{\sqrt{\pi}}(x-\sqrt{2}\,\mathrm{Re}\,z)^2\exp[-(x-\sqrt{2}\,\mathrm{Re}\,z)^2] \tag{42a}$$

$$|\langle x_b|\hat{z}_s\rangle|^2 = \frac{1}{2\sqrt{\pi}}\exp[-(x-\sqrt{2}\,\mathrm{Re}\,z)^2]. \tag{42b}$$

The entropy of x when the system is in the unitary state $|\hat{z}_s\rangle$ is given by

$$S_x(|\hat{z}_s\rangle) = -\int \mathrm{d}x\{|\langle x_f|\hat{z}_s\rangle|^2\ln|\langle x_f|\hat{z}_s\rangle|^2 + |\langle x_b|\hat{z}_s\rangle|^2\ln|\langle x_b|\hat{z}_s\rangle\rangle|^2\}. \tag{42c}$$

Insertion of the values found in (42a, b) for $|\langle x_f|\hat{z}_s\rangle|^2$, $|\langle x_b|\hat{z}_s\rangle|^2$ gives for this expression

$$\begin{aligned} S_x(|\hat{z}_s\rangle) &= -\int_{-\infty}^{+\infty}\mathrm{d}\eta\frac{1}{\sqrt{\pi}}\eta^2\exp(-\eta^2)\ln\left(\frac{1}{\sqrt{\pi}}\eta^2\exp(-\eta^2)\right) \\ &\quad -\int_{-\infty}^{+\infty}\frac{1}{2\sqrt{\pi}}\mathrm{d}\eta\exp(-\eta^2)\ln\left(\frac{1}{2\sqrt{\pi}}\exp(-\eta^2)\right) \\ &= \frac{2}{\sqrt{\pi}}\int_0^{\infty}\mathrm{d}\eta\,\eta^4\exp(-\eta^2) - \frac{4}{\sqrt{\pi}}\int_0^{\infty}\mathrm{d}\eta\,\eta^2\ln\eta\exp(-\eta^2) \\ &\quad +\frac{(\ln\pi+1)}{\pi^{1/2}}\int_0^{\infty}\eta^2\exp(-\eta^2)\,\mathrm{d}\eta + \frac{\ln 2\pi^{1/2}}{\pi^{1/2}}\int_0^{\infty}\mathrm{d}\eta\exp(-\eta^2) \\ &= \tfrac{1}{2}\ln\pi+\tfrac{3}{2}\ln 2+\tfrac{1}{2}C \end{aligned} \tag{42d}$$

where C is the Euler constant.

In a similar way we can evaluate the entropy of the momentum when the system is in the state $|z_s\rangle$.

The momentum eigenstates are given by

$$|p_f\rangle = \begin{pmatrix} |p\rangle \\ \cdot \end{pmatrix} \qquad |p_b\rangle = \begin{pmatrix} \cdot \\ |p\rangle \end{pmatrix}. \tag{43}$$

Then

$$S_p(|\hat{z}_s\rangle) = -\int \mathrm{d}p(|\langle p_f|\hat{z}_s\rangle|^2 \ln|\langle p_f|\hat{z}_s\rangle|^2 + |\langle p_b|\hat{z}_s\rangle|^2 \ln|\langle p_b|\hat{z}_s\rangle|^2) \tag{44a}$$

where

$$|\langle p_f|\hat{z}_s\rangle|^2 = \frac{1}{\pi^{1/2}}(p-\sqrt{2}\,\mathrm{Im}\,z)^2 \exp[-(p-\sqrt{2}\,\mathrm{Im}\,z)^2] \tag{44b}$$

and

$$|\langle p_b|\hat{z}_s\rangle|^2 = \frac{1}{2\pi^{1/2}} \exp[-(p-\sqrt{2}\,\mathrm{Im}\,z)^2]. \tag{44c}$$

After introduction of these values of $\{\langle p_f|\hat{z}_s\rangle;\ \langle p_s|\hat{z}_s\rangle\}$ into (20*a*) we obtain an expression identical to (42*d*), i.e.

$$S_p(|\hat{z}_s\rangle) = \tfrac{1}{2}\ln \pi + \tfrac{3}{2}\ln 2 + \tfrac{1}{2}C. \tag{44d}$$

The entropic uncertainty of $\{x, p\}$ when the system is set in $|\hat{z}_s\rangle$ turns out to be

$$U(x, p, |\hat{z}_s\rangle) \equiv S_x(|\hat{z}_s\rangle) + S_p(|\hat{z}_s\rangle) = 3\ln 2 + C + \ln \pi$$

$$\approx 2.7 + \ln \pi. \tag{44e}$$

Observe that neither $U(x, p, |\hat{z}_f\rangle)$ nor $U(x, p, |\hat{z}_s\rangle)$ depend upon z. They have a fixed value either for any fermionic supercoherent state $|z_f\rangle$ or for any pure supersymmetric state $|\hat{z}_s\rangle$.

4. Time evolution of $\{|z_f\rangle, |z_s\rangle\}$

Since $\Psi_{n>0} = \alpha\binom{|n\rangle}{\cdot} + \beta\binom{\cdot}{|n-1\rangle}$; $\Psi_0 = \binom{|0\rangle}{\cdot}$ it is immediately obvious that their time evolution is given by

$$\Psi_n(t) = \alpha\exp(-\mathrm{i}n\omega t)\begin{pmatrix} |n\rangle \\ \cdot \end{pmatrix} + \beta\exp(-\mathrm{i}n\omega t)\begin{pmatrix} \cdot \\ |n-1\rangle \end{pmatrix} \tag{45a}$$

$$\Psi_0(t) = \Psi_0 = \begin{pmatrix} |0\rangle \\ \cdot \end{pmatrix}. \tag{45b}$$

The evolution of $|z_f\rangle$ is easily obtained after (45*a*) and (45*b*)

$$|z_f(t)\rangle = \begin{pmatrix} |z\,\mathrm{e}^{-\mathrm{i}\omega t}\rangle \\ \cdot \end{pmatrix} = |(z\,\mathrm{e}^{-\mathrm{i}\omega t})_f\rangle \tag{46a}$$

while the time evolution of $|z_s\rangle$ takes the form

$$|z_s(t)\rangle = \exp(-\mathrm{i}\omega t)|(z\exp(-\mathrm{i}\omega t))_s\rangle. \tag{46b}$$

Both $|z_f(t)\rangle$ and $|z_s(t)\rangle$ retain their original structure. Consequently they do not spread along their time evolution keeping their respective uncertainties (24c)-(24e), (35a)-(35c) constant.

On the contrary the generic state (34) will have a modulation in its spreading (35e) and (35f) due to $\phi \to \phi + \omega t$. This property points towards showing that the really relevant supercoherent states are just $|z_f\rangle$ and $|z_s\rangle$.

For an arbitrary $|z\rangle = \sin\theta\, e^{i\varphi}|z_f\rangle + \cos\theta|z_s\rangle$ in the two-dimensional space $\{|z_f\rangle, |z_s\rangle\}$ its evolution will have the form

$$|z(t)\rangle = \exp(-i\omega t)\{\sin\theta \exp[i(\varphi+\omega t)]|(z\exp(-i\omega t))_f\rangle + \cos\theta|(z\exp(-i\omega t))_s\rangle\} \tag{46c}$$

showing an overall phase factor $\exp(-i\omega t)$, irrelevant in any physical calculation concerning the observables $\{x, x^2, p, p^2\}$ and a t dependence in the φ phase coefficient which will introduce time dependences in the computation of quantities like (35).

It is also interesting to obtain the mean value of the energy when the system is in the generic state $|z, \phi, \theta\rangle$. Using the properties quoted in (22) we have

$$\omega^{-1}\langle z, \theta, \varphi|H|z, \theta, \varphi\rangle = |z|^2 + \cos^2\theta - \frac{1}{\sqrt{2}}\mathrm{Re}\{z\, e^{i\varphi}\}\sin 2\theta. \tag{47}$$

This result shows that the generic states $|z, \theta, \phi\rangle$ may have energies quite far from $|z|^2$, unless $\theta = 0$ (and we are in the state $|z_s\rangle$) or $\theta = \pi/2$ (corresponding to $|z_f\rangle$). In these two cases the last term in (47) vanishes, giving rise to

$$\langle z_s|H|z_s\rangle = (|z|^2 + 1)\omega \tag{48a}$$

$$\langle z_f|H|z_f\rangle = |z|^2\omega \tag{48b}$$

respectively.

5. Discussion and comments

Using the simplest quantum mechanical ($N = 1$) supersymmetric system which is the natural generalisation of the standard quantum mechanical harmonic oscillator we have defined the supersymmetric annihilation operator A which transforms the $E_n = n\omega$ energy eigenspace into the E_{n-1} two-dimensional eigenspace.

The existence of this operator allowed us to define the candidate's supercoherent states as its eigenstates. Actually there exists a two-dimensional eigenspace parametrised by $|z, \theta, \phi\rangle$ having a natural orthogonal basis $\{|z_f\rangle, |z_s\rangle\}$ composed by a pure fermionic state and a pure supersymmetric state (in the sense that the mean value of the Klein operator when the system is set in $|z_s\rangle$ vanishes). They both have the same norm.

The sets $\{|z_f\rangle, |z_s\rangle\}$ constitute the right generalisation of the well known coherent states $|z\rangle$ of the harmonic oscillator. They both have a classical behaviour, do not spread in time and have a fixed uncertainty, which reaches the Heisenberg minimum for the set $\{|z_f\rangle\}$. Their energy has also a classical behaviour, particularly good for the pure fermionic supercoherent states. We have shown that the fermionic states $|z_f\rangle$ constitute the more classical states of the supersymmetric harmonic oscillator.

It seems natural to conjecture that for extended supersymmetric quantum mechanical systems there will always exist a finite set of supercoherent states which will have all the nice classical properties found in this model.

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